CENTER PROBLEM IN THE CENTER MANIFOLD FOR QUADRATIC DIFFERENTIAL SYSTEMS IN $\mathbb{R}^3$

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Abstract. Using tools of computer algebra based on the Gröbner basis theory we derive conditions for the existence of a center on a local center manifold for fifteen seven-parameter families of quadratic systems on $\mathbb{R}^3$. To obtain the results we use modular arithmetics.

1. Introduction and statement of the main results

In this paper we consider three-dimensional polynomial systems of the form

$$\dot{u} = v + P(u,v,w), \qquad \dot{v} = -u + Q(u,v,w), \qquad \dot{w} = -w + R(u,v,w), \tag{1}$$

where P, Q and R are polynomials of degree at most N without constant or linear terms. The linear part of system (1) has one non–zero and two pure imaginary eigenvalues. Moreover we assume that system (1) has an isolated fixed point at the origin. It is well-known that system (1) admits a local center manifold at the origin. We aim to investigate the local flow on a neighborhood of the origin on the local center manifold in order to distinguish whether the origin in the center manifold is a center or a focus.

This problem is a generalization to $\mathbb{R}^3$ of the classical center-focus problem for differential systems in the plane and is one of the most famous problems in qualitative theory of ordinary differential equations. The center-focus problem for polynomial differential systems in $\mathbb{R}^2$ goes back to Poincaré [15, 16] and it has been intensively investigated by many authors, see for instance [7, 8, 12] and the references therein.

There are many systems in $\mathbb{R}^3$ that arise naturally in sciences and engineering that possess a fixed point at which the linear part has one negative and two purely imaginary eigenvalues, hence which can be placed into the form (1). Among them the Rikitake system (Earth's magnetic field) [17, 18], the Hide–Skeldon–Acheson Dynamo [18, 19] and the Moon–Rand system (flexible structures) [20] are examples of systems in $\mathbb{R}^3$ with such linear part and with quadratic nonlinearities. In fact the Moon–Rand system is a system in $\mathbb{R}^3$ of the form

$$\dot{u} = v, \quad \dot{v} = -u - uw, \quad \dot{w} = -\lambda w + au^2 + buv + cv^2. \tag{2}$$

with λ, a, b and $c \in \mathbb{R}$. This system is studied in [14] where the center problem of the center manifold is solved for any λ, a, b and $c \in \mathbb{R}$.

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In fact few results are known providing center conditions in dimensions greater than two. For instance, recently the conjecture of Mello and Coelho [21] concerning the existence of centers on local center manifolds at equilibria for the Lü system

$$\dot{x} = a(y - x), \quad \dot{y} = cy - xz, \quad \dot{z} = -bz + xy,$$

has been confirmed, see [22]. In [5] the authors give a method to solve this problem and show that for each fixed value of the non-zero real eigenvalue the set of systems having a center on the local center manifold at the origin corresponds to a variety in the space of admissible coefficients. In [13, 14, 25] the authors apply the techniques developed in [5] to several families of systems with quadratic or cubic higher order terms.

In [25] the class of three-dimensional differential systems of the form

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= -u + a_1u^2 + a_2uv + a_3uw + a_4v^2 + a_5vw + a_6w^2, \\ \dot{w} &= -w + c_1u^2 + c_2uv + c_3v^2 \end{aligned} \tag{3}$$

where $a_i, c_j \in \mathbb{R}$ is studied. Indeed it is studied the center problem in the local center manifold. System (3) is a generalization of the Moon–Rand system (2) for the case $\lambda = 1$. The case $\lambda \neq 1$ is much more difficult as you can see in [14, 25].

We recall that system (3) admits a local center manifold at the origin due to the Center Manifold Theorem, see for instance [2]. We write the center manifold as $w = \phi(u, v)$ for some smooth function ϕ. More specifically in [25] it is given the necessary and sufficient conditions for the existence of a center on the local center manifold for the following six-parameter families of system (3):

(a) $a_3 = a_5 = a_6 = 0$;
(b) $a_2 = a_4 = a_5 = 0$;
(c) $a_1 = a_2 = a_3 = 0$.

In this paper we improve the results obtained in [25] since we are able to obtain the necessary and sufficient conditions (with very high probability, see below Conjecture 16) for the existence of a center on the local center manifold of the following fifteen seven-parameter families of system (3):

(a) $a_1 = a_2 = 0$; (b) $a_1 = a_3 = 0$; (c) $a_1 = a_4 = 0$; (d) $a_1 = a_5 = 0$;
(e) $a_1 = a_6 = 0$; (f) $a_2 = a_3 = 0$; (g) $a_2 = a_4 = 0$; (h) $a_2 = a_5 = 0$;
(i) $a_2 = a_6 = 0$; (j) $a_3 = a_4 = 0$; (k) $a_3 = a_5 = 0$; (l) $a_3 = a_6 = 0$;
(m) $a_4 = a_5 = 0$; (n) $a_4 = a_6 = 0$; (o) $a_5 = a_6 = 0$.

More precisely, the main results of this paper are the following.

Theorem 1. *System* (3) *with* $a_1 = a_2 = 0$ *admits a center on the local center manifold if one of the following conditions holds:*

(a.1) $c_1 = c_2 = c_3 = 0$;
(a.2) $a_5 = c_3 = 2c_1 - c_2 = 0$;
(a.3) $a_3 = a_5 = a_6 = 0$.

Theorem 2. *System* (3) *with* $a_1 = a_3 = 0$ *admits a center on the local center manifold if one of the following conditions holds:*

(b.1) $a_4 = c_1 = c_2 = c_3 = 0$;
(b.2) $a_2 = c_1 = c_2 = c_3 = 0$;

(b.3) $2a_2a_4+a_5c_2 = a_4a_5^2-a_2^2a_6 = 2a_4^2a_5+a_2a_6c_2 = 4a_4^3-a_6c_2^2 = 2a_2^3a_6+a_5^3c_2 = 2c_1-c_2 = c_3 = 0$;
(b.4) $a_2 = a_5 = 2c_1-c_2 = c_3 = 0$;
(b.5) $a_4 = a_5 = a_6 = 0$;
(b.6) $a_2 = a_5 = a_6 = 0$.

Theorem 3. *System* (3) *with* $a_1 = a_4 = 0$ *admits a center on the local center manifold if one of the following conditions holds:*

(c.1) $c_1 = c_2 = c_3 = 0$;
(c.2) $a_5 = a_6 = 2c_1-c_2 = c_3 = 0$;
(c.3) $a_2 = a_5 = 2c_1-c_2 = c_3 = 0$;
(c.4) $a_3 = a_5 = a_6 = 0$.

Theorem 4. *System* (3) *with* $a_1 = a_5 = 0$ *admits a center on the local center manifold if one of the following conditions holds:*

(d.1) $a_4 = c_1 = c_2 = c_3 = 0$;
(d.2) $a_4 = a_6 = 2c_1-c_2 = c_3 = 0$;
(d.3) $a_2 = 2c_1-c_2 = c_3 = 0$;
(d.4) $a_3 = a_4 = a_6 = 0$;
(d.5) $a_2 = a_3 = a_6 = 0$.

Note that conditions (d.1) and (d.2) are conditions (c.1) and (c.2) of Theorem 3. Conditions (d.3) and (d.5) are conditions (a.2), (a.3) of Theorem 1 and condition (d.4) is condition (b.5) of Theorem 2.

Theorem 5. *System* (3) *with* $a_1 = a_6 = 0$ *admits a center on the local center manifold if one of the following conditions holds:*

(e.1) $a_4 = c_1 = c_2 = c_3 = 0$;
(e.2) $a_2 = c_1 = c_2 = c_3 = 0$;
(e.3) $a_4 = a_5 = 2c_1-c_2 = c_3 = 0$;
(e.4) $a_2 = a_5 = 2c_1-c_2 = c_3 = 0$;
(e.5) $a_3 = a_4 = a_5 = 0$;
(e.6) $a_2 = a_3 = a_5 = 0$.

Note that conditions (e.1) and (e.3) are conditions (c.1) and (c.2) of Theorem 3. Conditions (e.2), (e.4) and (e.6) are conditions (a.1), (a.2), (a.3) of Theorem 1 and condition (e.5) is condition (b.5) of Theorem 2.

Theorem 6. *System* (3) *with* $a_2 = a_3 = 0$ *admits a center on the local center manifold if one of the following conditions holds:*

(f.1) $a_5 = a_6 = 0$;
(f.2) $a_5 = 2c_1-c_2 = c_3 = 0$;
(f.3) $c_1 = c_2 = c_3 = 0$.

Theorem 7. *System* (3) *with* $a_2 = a_4 = 0$ *admits a center on the local center manifold if one of the following conditions holds:*

(g.1) $a_5 = 2c_1-c_2 = c_3 = 0$;
(g.2) $c_1 = c_2 = c_3 = 0$;
(g.3) $a_3 = a_5 = a_6 = 0$.

Theorem 8. *System* (3) *with* $a_2 = a_5 = 0$ *admits a center on the local center manifold if one of the following conditions holds:*

(h.1) $a_3 = a_6 = 0$;
(h.2) $a_1 = a_3 = a_4 = c_1 = c_2 = c_3 = 0$;
(h.3) $2c_1 - c_2 = c_3 = 0$.

Theorem 9. *System* (3) *with* $a_2 = a_6 = 0$ *admits a center on the local center manifold if one of the following conditions holds:*

(i.1) $c_1 = c_2 = c_3 = 0$;
(i.2) $a_5 = 2c_1 - c_2 = c_3 = 0$;
(i.3) $a_3 = a_5 = 0$.

Theorem 10. *System* (3) *with* $a_3 = a_4 = 0$ *admits a center on the local center manifold if one of the following conditions holds:*

(j.1) $a_2 = c_1 = c_2 = c_3 = 0$;
(j.2) $a_1 = c_1 = c_2 = c_3 = 0$;
(j.3) $a_6 = 2a_1a_2 + a_5c_2 = 2c_1 - c_2 = c_3 = 0$;
(j.4) $a_2 = a_5 = a_6 = 0$;
(j.5) $a_1 = a_5 = a_6 = 0$;
(j.6) $a_2 = a_5 = c_3 = 2c_1 - c_2 = 0$.

Theorem 11. *System* (3) *with* $a_3 = a_5 = 0$ *admits a center on the local center manifold if one of the following conditions holds:*

(k.1) $a_2 = a_6 = 0$;
(k.2) $a_2 = 2c_1 - c_2 = c_3 = 0$;
(k.3) $a_1 + a_4 = a_6 = 0$;
(k.4) $a_1 + a_4 = c_1 = c_2 = c_3 = 0$.

Theorem 12. *System* (3) *with* $a_3 = a_6 = 0$ *admits a center on the local center manifold if one of the following conditions holds:*

(l.1) $a_2 = a_5 = 0$;
(l.2) $a_5 = a_1 + a_4 = 0$;
(l.3) $a_2 = c_1 = c_2 = c_3 = 0$;
(l.4) $a_1 + a_4 = c_1 = c_2 = c_3 = 0$;
(l.5) $2a_1a_2 - a_5c_2 = a_4 = 2c_1 - c_2 = c_3 = 0$.

Theorem 13. *System* (3) *with* $a_4 = a_5 = 0$ *admits a center on the local center manifold if one of the following conditions holds:*

(m.1) $a_1 = c_1 = c_2 = c_3 = 0$;
(m.2) $a_1 = a_6 = 2c_1 - c_2 = c_3 = 0$;
(m.3) $a_2 = 2c_1 - c_2 = c_3 = 0$;
(m.4) $a_2 = a_3 = a_6 = 0$;
(m.5) $a_1 = a_3 = a_6 = 0$.

Note that conditions (m.1), (m.2) and (m.5) are conditions (c.1), (c.2) and (c.4) of Theorem 3 and conditions (m.3) and (m.4) are conditions (g.1) and (g.3) of Theorem 7.

Theorem 14. *System* (3) *with* $a_4 = a_6 = 0$ *admits a center on the local center manifold if one of the following conditions holds:*

(n.1) $a_2 = c_1 = c_2 = c_3 = 0$;
(n.2) $a_1 = c_1 = c_2 = c_3 = 0$;
(n.3) $a_2 = a_5 = 2c_1 - c_2 = c_3 = 0$;

(n.4) $a_1 = a_5 = 2c_1 - c_2 = c_3 = 0$;
(n.5) $a_3 = 2a_1a_2 - a_5c_2 = 2c_1 - c_2 = c_3 = 0$;
(n.6) $a_2 = a_3 = a_5 = 0$;
(n.7) $a_1 = a_3 = a_5 = 0$.

Note that conditions (n.1), (n.3) and (n.6) are conditions (g.2), (g.1) and (g.3) of Theorem 7. Conditions (n.2), (n.4) and (n.7) are conditions (c.1), (c.2) and (c.4) of Theorem 3, and condition (n.5) is condition (j.3) of Theorem 10.

Theorem 15. *System* (3) *with* $a_5 = a_6 = 0$ *admits a center on the local center manifold if one of the following conditions holds:*

(o.1) $a_2 = a_3 = 0$;
(o.2) $a_1 + a_4 = a_3 = 0$;
(o.3) $a_1 + a_4 = c_1 = c_2 = c_3 = 0$;
(o.4) $a_2 = 2c_1 - c_2 = c_3 = 0$;
(o.5) $a_1 = a_4 = 2c_1 - c_2 = c_3 = 0$.

The proof of Theorems 1–3, 6–12 and Theorem 15 are given in section 4.

The computation of the focus quantities and the decomposition of the variety of the ideal that they generate were used to obtain the necessary conditions, see section 3. Since this decomposition of the variety was performed using modular computations the obtained cases represent the complete list of center conditions only with very high probability. So, it is not proved that the center conditions given in Theorems 1–15 are necessary and sufficient to have a center on the local center manifold and the following conjecture remains open.

Conjecture 16. *The center conditions given in Theorems 1–15 are necessary and sufficient to have a center on the local center manifold.*

The problem that we present might be completely solved when the computing power facilities will be better than the current ones. In fact cases that were unthinkable a few years ago can now be addressed. These type of problems can be regarded as a testing ground for the algorithms developed by the symbolic computation packages because for each necessary condition the sufficiency must be proved confirming that the center conditions are correct. However, as will be seen through the paper, the center conditions are simple relationships between the parameters of the system. This raises an interesting theoretical problem which is to obtain these conditions without going through the existing methods to find the necessary conditions.

In the next section we provide definitions and general techniques that will be necessary for the proof of our main theorems. We recall that in [14] the authors provide a Mathematica code for the automatic computation of the coefficients of the lowest order terms in the expansion of the center manifold in a neighborhood of the origin and for the computation of the first focus quantities. The method used in [14] is the complexification of system (1). Here, in this paper, our approach is a little bit different and we work directly with the real system (3). We first compute an approximation of the center manifold and we substitute this approximation into the second equation of system (3) to compute the Poincaré-Liapunov quantities of the two-dimensional system formed by the first two equations of system (3).

2. Preliminary results

We briefly discuss the procedure to study the center problem on a center manifold for the three-dimensional differential system (1). We will denote by $\mathcal{X}$ the corresponding vector field

$$\mathcal{X} = (-v+P)\frac{\partial}{\partial u} + (-u+Q)\frac{\partial}{\partial v} + (-w+R)\frac{\partial}{\partial w}.$$

A *local analytic first integral* I of system (1) is a nonconstant differentiable function that is analytic and constant on the trajectories of (1), i.e.,

$$\mathcal{X}I = (-v+P)\frac{\partial I}{\partial u} + (-u+Q)\frac{\partial I}{\partial v} + (-w+R)\frac{\partial I}{\partial w} \equiv 0.$$

A *formal first integral* I for system (1) is a nonconstant formal power series in the variables u, v, w that satisfies $\mathcal{X}I = 0$.

One of the main tools for knowing whether a system has a center on a center manifold is the following theorem whose proof is given in [5].

Theorem 17. *The following statements are equivalent for system* (1)*.*

- (i) *The origin is a center for the vector field $\mathcal{X}$ restricted to the center manifold.*
- (ii) *The system admits a local analytic first integral Φ. It can always be chosen of the form $\Phi = u^2 + v^2 + \cdots$ in a neighborhood of $\mathbb{R}^3$ (the dots indicate higher order terms)*
- (iii) *The system admits a formal first integral.*

In fact Theorem 17 is an improvement of the Lyapunov Center Theorem proved in [1, section 13].

3. Necessary conditions to have a center

The method used in [14] is the computation of the formal first integral of Statement (ii). The authors introduce the complex variables $x = u + iv$, $y = \bar{x} = u - iv$. Then the first two equations are equivalent to a single equation in complex variables given by $\dot{x} = ix + X(x, y, w)$, where X is the sum of homogeneous polynomials of degrees between 2 and N. Moreover once we have the differential equation for x we also have the differential equation for $\bar{x}$. Therefore, the complexification of system (1) is given by

$$\begin{aligned}
\dot{x} &= ix + \sum_{j+k+l=2}^{N} a_{jkl} x^j y^k z^l, \\
\dot{y} &= -iy + \sum_{j+k+l=2}^{N} b_{jkl} x^j y^k z^l, \\
\dot{w} &= -w + \sum_{j+k+l=2}^{N} c_{jkl} x^j y^k w^l,
\end{aligned} \tag{4}$$

where $b_{jkl} = \overline{a}_{kjl}$ and c_{jkl} are such that $\sum_{j+k+l=2}^{N} c_{jkl} x^j \bar{x}^k w^l$ is real for all $x \in \mathbb{C}$ and all $w \in \mathbb{R}$.

Now we investigate the existence of a first integral $\Psi = \Psi(x, y, w)$ for system (4), which is equivalent to the existence of a first integral $\Phi = \Phi(u, v, w)$ for system

(1). Note that Ψ will be given in the form

$$\Psi(x,y,w) = xy + \sum_{k+l+j\geq 3} d_{klj}x^k y^l w^j. \tag{5}$$

Let $\tilde{\mathcal{X}}$ denote the vector field associated to system (4) in $\mathbb{C}^3$. We note that Ψ satisfies $\tilde{\mathcal{X}}\Psi \equiv 0$. Using that Ψ has the form (5) we get (see [5] for details)

$$\tilde{\mathcal{X}}\Psi(x,y,z) = g_{220}(xy)^2 + g_{330}(xy)^3 + \cdots$$

The quantities g_{kk0} are called the *focus quantities* of system (4). The vanishing of all the focus quantities is necessary and sufficient for the existence of a formal first integral, see [5].

However the complexification of the real system (1) is not necessary. The method used in this work permits to reduce the center problem on the center manifold for the real system (3) to a classical center problem of an analytic differential system in the plane. Here the computation of the center conditions is done in the following way: First we construct a polynomial approximation of any center manifold at the origin up to certain degree. Thus we express the center manifold as

$$w = h(u,v) = h_{20}u^2 + h_{11}uv + h_{02}v^2 + \cdots,$$

where h_{ij} are parameters to be determined. The coefficients h_{ij} are found by equating coefficients in the expression that determines the center manifold,

$$\frac{\partial h}{\partial u}\dot{u} + \frac{\partial h}{\partial v}\dot{v} = -h(u,v) + c_1u^2 + c_2uv + c_3v^2.$$

Next we consider the two-dimensional differential system (3) where w is substituted by the computed approximation up to a certain degree, obtaining the two-dimensional differential system

$$\dot{u} = v, \quad \dot{v} = -u + a_1u^2 + a_2uv + a_3uh(u,v) + a_4v^2 + a_5vh(u,v) + a_6h(u,v)^2. \tag{6}$$

For system (6) we apply the classical method of changing to polar coordinates $u = r\cos\theta$, $v = r\sin\theta$, and propose a formal first integral of the form $H = r^2/2 + \cdots$ to find the obstructions to its existence which give the so-called *Poincaré-Liapunov constants or Poincaré-Liapunov quantities* V_{2k}, that is, $\dot{H} = \sum_{k=2} V_{2k}\, r^{2k}$, see for instance [8]. In fact these Poincaré-Liapunov quantities are polynomials in the parameters of system (6).

The vanishing of a finite number of focus quantities g_{kk0} (using the method developed in [5]) or Poincaré-Liapunov quantities V_{2k} gives only the necessary conditions for the existence of a center since it is not known a priori how many of these Poincaré-Liapunov quantities are needed to generate the full ideal $B := \langle V_{2k} : k \in \mathbb{N}\rangle$. The computation of the Poincaré-Liapunov quantities described above requires the use of a computer and a symbolic computation package. We used an algorithm developed in a Mathematica code.

To verify if the number of Poincaré-Liapunov quantities that we computed a priori is enough to generate the full ideal we proceed as follows: Let B_i be the ideal generated only by the first i focus quantities, i.e., $B_{2i} = \langle V_4, \ldots, V_{2i}\rangle$. We want to determine s so that $V(B) = V(B_s)$, being V the variety of the ideals B and B_s, respectively. Using the Radical Membership Test [24] we can find when the computation stabilizes in the sense that $\sqrt{B_{s-1}} \subset \sqrt{B_s}$ but $\sqrt{B_s} = \sqrt{B_{s+1}}$. It is clear that $V(B) \subset V(B_s)$. However to verify the opposite inclusion we first need to obtain the irreducible decomposition of the variety of $V(B_s)$ and check that any point of each component corresponds to a system having a center at the local center manifold.

The computation of the irreducible decomposition of $V(B_s)$ can be done decomposing the radical $\sqrt{B_s}$ into an intersection of prime ideals. This requires a more specialized computer algebra package such as Singular [9]. We perform our computations with the routine `minAssGTZ` of Singular (library primedec.lib [3]) in order to find the minimal associated primes of the polynomial ideal by means of the Gianni-Trager-Zacharias method [6]. Some definitions and algorithms of computational algebra that we will use here can be reminded in [10].

Therefore, to check if we have enough focus quantities, we decompose the radical $\sqrt{B_j}$ into an intersection of prime ideals, to which correspond the irreducible components of $V(B_j)$. The generators of these prime ideals give a finite set of conditions to have a center on the local center manifold. If these conditions for each component can be proved to be sufficient then we have the characterization of the existence of a center on the local center manifold.

Due to its complexity, we are not able to compute the decomposition over the rational field. Hence we use modular arithmetics. In fact the decomposition is obtained over characteristic 32003. To go back to the rational numbers we use the following rational reconstruction algorithm of Wang et al. [26] (in the algorithm, $\lfloor . \rfloor$ stands for the floor function).

1. $u = (u_1, u_2, u_3) := (1, 0, m), v = (v_1, v_2, v_3) := (1, 0, c)$
2. While $\sqrt{m/2} \leq v_3$ do $\{q := \lfloor u_3/v_3 \rfloor, r := u - qv, u := v, v := r\}$
3. If $|v_2| \geq \sqrt{m/2}$ then error ()
4. Return v_3, v_2

Given an integer c and a prime p, the algorithm produces integers v_2 and v_3 such that $v_3/v_2 \equiv c (\bmod\, p)$. Such a number v_3/v_2 need not exist. If this is the case, then the algorithm returns "error ()". As we have used modular arithmetics we must check if the decomposition is complete i.e., if no component is lost. In order to do that let P_i denote the polynomials defining each component. Using the instruction `intersect` of Singular we compute the intersection $P = \cap_i P_i = \langle p_1, \ldots, p_m \rangle$. By the Strong Hilbert Nullstellensatz (see for instance [24]) to check whether $V(B_j) = V(P)$ it is sufficient to check if the radicals of the ideals are the same, that is, if $\sqrt{B_j} = \sqrt{P}$. Computing over characteristic 0 reducing Gröbner bases of ideals $\langle 1 - wV_{2k}, P : V_{2k} \in B_j \rangle$ we find that each of them is $\{1\}$. By the Radical Membership Test this implies that $\sqrt{B_j} \subseteq \sqrt{P}$. To check the opposite inclusion, $\sqrt{P} \subseteq \sqrt{B_j}$ it is sufficient to check that

$$\langle 1 - wp_k, B_j : k = 1, \ldots, m \rangle = \langle 1 \rangle. \tag{7}$$

Trying to check (7) with the Radical Membership Test we were not able to complete computations working in the field of characteristic zero. However we have checked that (7) holds in several polynomial rings over fields of finite characteristic. It means that (7) holds with high probability and, therefore also that $V(B_j) = V(P)$ holds with high probability, see [23].

For example the first nonzero Poincaré-Liapunov quantity for system (3) with $a_1 = a_2 = 0$ is

$$V_4 = \frac{1}{40}(-2a_3c_1 + 9a_5c_1 + a_3c_2 - 2a_5c_2 + 2a_3c_3 + 11a_5c_3).$$

Due to their size we do not present the next six Poincaré-Liapunov quantities, but they can be easily computed using the method described in this section. A decomposition over a field of characteristic zero of the radical of the ideal B_6 into an intersection of prime ideals is computationally infeasible. However the decomposition is possible over the finite field of characteristic 32003, and after the rational reconstruction it is given by

$$J_1 = \langle c_1, c_2, c_3\rangle; \quad J_2 = \langle a_5, c_3, 2c_1 - c_2\rangle; \quad J_3 = \langle a_3, a_5, a_6\rangle,$$

and other complicated prime ideals that always have a_5 and a_3 as generators. This computation suggests that system (3) with $a_1 = a_2 = 0$ has a center if 1) $c_1 = c_2 = c_3 = 0$, or 2) $a_5 = c_3 = 2c_1 - c_2 = 0$, or 3) $a_3 = a_5 = 0$ with more conditions that need to be satisfied. With this hint we can verify the necessity of these three conditions as follows. If $a_5 \neq 0$ then a linear change of the form $u = U$, $v = V$ and $w = kW$ preserves the form of system (3) but with $a_5 = 1$. The decomposition of the radical of the ideal B_6 with $a_5 = 1$ computed over a field of characteristic zero gives

$$\langle c_1, c_2, c_3\rangle; \qquad \langle c_1 + c_3, 2a_3c_3 + c_2, c_2^2 + 4c_3^2, a_3c_2 - 2c_3, a_3^2 + 1\rangle.$$

Since we are concerned with real parameters, the decomposition of the radical of the ideal B_6 when $a_5 = 1$ is $\langle c_1, c_2, c_3\rangle$. In a similar fashion we show that if $a_3 \neq 0$ then with a linear change of variables we can always consider $a_3 = 1$. The decomposition of the radical of the ideal B_6 with $a_3 = 1$ computed over a field of characteristic zero gives

$$\langle c_1, c_2, c_3\rangle; \quad \langle a_5, c_3, 2c_1 - c_2\rangle; \quad \langle c_1 + c_3, 2a_5c_3 + c_2, c_2^2 + 4c_3^2, a_5c_2 - 2c_3, a_5^2 + 1\rangle,$$

which implies that the decomposition of the radical of the ideal B_6 when $a_3 = 1$ is given by the ideals J_1 and J_2 since we consider only real parameters. Finally the case $a_1 = a_2 = a_3 = a_5 = 0$ is a particular case studied in [25] that gives the unique case $\langle a_6\rangle$ giving the ideal J_3. Hence we obtain the conditions given in Theorem 1.

4. Appendix

Once we have described the method of procedure to prove necessity, we recall a result that will be used during the appendix of the paper to prove sufficiency. The following result was given by Dulac in [4] and Kapteyn in [11].

Theorem 18 (Quadratic Center Theorem)**.** *System*

$$\dot{u} = -v - bu^2 - (B + 2c)uv - dv^2, \qquad \dot{v} = u + au^2 + (A + 2b)uv + cv^2, \tag{8}$$

has a center at the origin if and only if at least one of the following three conditions hold:

(i) $a + c = b + d = 0$;
(ii) $A(a + c) = B(b + d)$ *and* $aA^3 - (3b + A)A^2B + (3c + B)AB^2 - dB^3 = 0$;
(iii) $A + 5d + 5b = B + 5a + 5c = ac + bd + 2a^2 + 2d^2 = 0$.

The method to find the necessary conditions and to prove that these necessary conditions are complete with high probability has been given in the previous section. Therefore in the following we only prove the sufficiency of the results. We will do it for each theorem.

Proof of Theorem 1. We consider system (3) with the conditions of Theorem 1 that is we consider system

$$
\begin{aligned}
\dot u &= v,\\
\dot v &= -u + a_3 uw + a_4 v^2 + a_5 vw + a_6 w^2,\\
\dot w &= -w + c_1 u^2 + c_2 uv + c_3 v^2.
\end{aligned} \tag{9}
$$

We start with case (a.1). System (9) with the conditions of Theorem 1 (a.1) becomes

$$
\begin{aligned}
\dot u &= v,\\
\dot v &= -u + a_3 uw + a_4 v^2 + a_5 vw + a_6 w^2,\\
\dot w &= -w.
\end{aligned} \tag{10}
$$

It is clear that $w = 0$ is an invariant algebraic surface for system (10) with cofactor $K = -1$ and it is a center manifold for system (10). Restricting it to the center manifold we get

$$
\dot u = v, \qquad \dot v = -u + a_4 v^2. \tag{11}
$$

Note that this system is system (8) with $b = d = 0$, $a = 0$, $c = a_4$ $A = 0$ and $B = -2a_4$. Hence, in view of Theorem 18 (ii) it has a center.

System (9) with the conditions of Theorem 1 (a.2) reads for

$$
\begin{aligned}
\dot u &= v,\\
\dot v &= -u + a_3 uw + a_4 v^2 + a_6 w^2,\\
\dot w &= -w + c_1 u^2 + 2c_1 uv.
\end{aligned} \tag{12}
$$

It is easy to check by direct computations that

$$
F = w - c_1 u^2 = 0, \tag{13}
$$

is an invariant algebraic surface for system (12) with cofactor $K = -1$. Since $F = 0$ is tangent to $w = 0$, it is a center manifold for system (12). Now we introduce the change of variables

$$
U = u, \quad V = v, \quad W = w - c_1 u^2. \tag{14}
$$

In these new variables system (12) becomes

$$
\begin{aligned}
\dot U &= V,\\
\dot V &= -U + a_3 U(W + c_1 U^2) + a_4 V^2 + a_6 (W + c_1 U^2)^2,\\
\dot W &= -W.
\end{aligned}
$$

The restriction on the center manifold $W = 0$ give rise to the system

$$
\dot U = V, \qquad \dot V = -U + a_3 c_1 U^3 + a_4 V^2 + a_6 c_1^2 U^4.
$$

Note that this systems is invariant by the symmetry $(U, V, t) \mapsto (U, -V, -t)$ and thus it admits a center on the local center manifold at the origin.

System (9) with the conditions of Theorem 1 (a.3) becomes

$$\begin{aligned}\dot u &= v,\\ \dot v &= -u + a_4 v^2,\\ \dot w &= -w + c_1 u^2 + 2c_1 uv.\end{aligned}$$

The center manifold is given by $w = \phi(u, v)$ for some analytic function ϕ. Indeed $w = c_1 u^2$. Restricting to this manifold we get that on the center manifold the dynamics is given by (11) and has a center at the origin due to Theorem 18 (ii). This concludes the proof. □

Proof of Theorem 2. We consider system (3) with the conditions of Theorem 2 that is we consider system

$$\begin{aligned}\dot u &= v,\\ \dot v &= -u + a_2 uv + a_4 v^2 + a_5 vw + a_6 w^2,\\ \dot w &= -w + c_1 u^2 + c_2 uv + c_3 v^2.\end{aligned} \tag{15}$$

Note that conditions (b.2), (b.4) and (b.6) are conditions (a.1), (a.2) and (a.3) of Theorem 1, respectively. Hence we will only proof conditions (b.1), (b.3) and (b.5). We start with case (b.1). System (15) with the conditions of Theorem 2 (b.1) becomes

$$\begin{aligned}\dot u &= v,\\ \dot v &= -u + a_2 uv + a_5 vw + a_6 w^2,\\ \dot w &= -w.\end{aligned} \tag{16}$$

It is clear that $w = 0$ is an invariant algebraic surface for system (16) with cofactor $K = -1$ and it is a center manifold for system (16). Restricting it to the center manifold we get

$$\dot u = v, \qquad \dot v = -u + a_2 uv. \tag{17}$$

Note that this system is system (8) with $b = d = 0$, $a = c = 0$, $A = a_2$ $A = B = 0$. Hence, in view of Theorem 18 (i) it has a center.

System (15) with the conditions of Theorem 2 (b.3) becomes

$$\begin{aligned}\dot u &= v,\\ \dot v &= -u + a_2 uv + a_4 v^2 + a_5 vw + a_6 w^2,\\ \dot w &= -w + c_1 u^2 + 2c_1 uv,\end{aligned} \tag{18}$$

where $2a_2a_4 + a_5c_2 = a_4a_5^2 - a_2^2a_6 = 2a_4^2a_5 + a_2a_6c_2 = 4a_4^3 - a_6c_2^2 = 2a_2^3a_6 + a_5^3c_2 = 0$. Note that F given in (13) is an invariant algebraic surface for system (18) with cofactor $K = -1$. Since $F = 0$ is tangent to $w = 0$, it is a center manifold for system (18). Now we introduce the change of variables in (14). In these new variables system (18) becomes

$$\begin{aligned}\dot U &= V,\\ \dot V &= -U + a_2 UV + a_4 V^2 + a_5 V(W + c_1 U^2) + a_6 (W + c_1 U^2)^2,\\ \dot W &= -W.\end{aligned}$$

The restriction on the center manifold $W=0$ give rise to the system

$$\dot U = V, \qquad \dot V = -U + a_2UV + a_4V^2 + a_5c_1VU^2 + a_6c_1^2U^4, \tag{19}$$

where $2a_2a_4+a_5c_2 = a_4a_5^2-a_2^2a_6 = 2a_4^2a_5+a_2a_6c_2 = 4a_4^3-a_6c_2^2 = 2a_2^3a_6+a_5^3c_2 = 0$. If $a_2=0$ we have $a_5c_2=0$. If $a_5=0$ then we obtain a particular case of case (b.4). If $c_2=0$ we are in the particular case (b.2). Hence we can assume $a_2 \neq 0$. In this case from the condition $2a_2a_4+a_5c_2=0$ we isolate $a_4=-a_5c_1/a_2$ recalling that $c_2=2c_1$. The condition $a_4a_5^2-a_2^2a_6=0$ takes the form $a_5^3c_1+a_2^3a_6=0$. We isolate $a_6=-a_5^3c_1/a_2^3$ and all the other conditions vanish. System (19) takes the form

$$\dot U = V, \qquad \dot V = -U + a_2UV - \frac{a_5c_1}{a_2}V^2 + a_5c_1VU^2 - \frac{a_5^3c_1^3}{a_2^3}U^4. \tag{20}$$

System (20) has the inverse integrating factor

$$\mathcal{V} = 1 - \frac{2a_5c_1U}{a_2} - a_2V + \frac{3a_5^2c_1^2U^2}{a_2^2} + a_5c_1UV + \frac{a_5^2c_1^2V^2}{a_2^2} - \frac{2a_5^3c_1^3U^3}{a_2^3} - \frac{a_5^2c_1^2U^2V}{a_2} + \frac{a_5^4c_1^4U^4}{a_2^4}.$$

Hence, as $\mathcal{V}(0,0)\neq 0$ system (20) has a center in at the origin, see [7].

System (15) with the conditions of Theorem 2 (b.5) becomes

$$\begin{aligned}
\dot u &= v,\\
\dot v &= -u + a_2uv,\\
\dot w &= -w + c_1u^2 + c_2uv + c_3v^2,
\end{aligned}$$

The center manifold is given by $w=\phi(u,v)$ for some analytic function ϕ. Restricting to this manifold we get that on the center manifold the dynamics is given by system (17) and therefore it has a center. □

Proof of Theorem 3. We consider system (3) with the conditions of Theorem 3 that is we consider system

$$\begin{aligned}
\dot u &= v,\\
\dot v &= -u + a_2uv + a_3uw + a_5vw + a_6w^2,\\
\dot w &= -w + c_1u^2 + c_2uv + c_3v^2.
\end{aligned} \tag{21}$$

Note that conditions (c.3) and (c.4) are conditions (a.2) of Theorem 1 and condition (b.5) of Theorem 2, respectively. Hence we will only proof conditions (c.1) and (c.2). System (21) with the conditions of Theorem 3 (c.1) becomes

$$\begin{aligned}
\dot u &= v,\\
\dot v &= -u + a_2uv + a_3uw + a_5vw + a_6w^2,\\
\dot w &= -w.
\end{aligned} \tag{22}$$

It is clear that $w=0$ is an invariant algebraic surface for system (22) with cofactor $K=-1$ and it is a center manifold for system (22). Restricting it to the center manifold we get (17) which in view of Theorem 18 (i) it has a center.

System (21) with the conditions of Theorem 3 (c.2) becomes

$$\begin{aligned}
\dot u &= v,\\
\dot v &= -u + a_2uv + a_3uw,\\
\dot w &= -w + c_1u^2 + 2c_1uv.
\end{aligned} \tag{23}$$

We observe that F given in (13) is an invariant algebraic surface for system (23) with cofactor $K = -1$. Since $F = 0$ is tangent to $w = 0$, it is a center manifold for system (23). Now we introduce the change of variables given in (14). In these new variables system (23) becomes

$$\begin{aligned}\dot{U} &= V,\\ \dot{V} &= -U + a_2UV + a_3U(W + c_1U^2),\\ \dot{W} &= -W.\end{aligned}$$

The restriction on the center manifold $W = 0$ give rise to the system

$$\dot{U} = V, \qquad \dot{V} = -U + a_2UV + a_3c_1U^3.$$

Note that this system is invariant by the symmetry $(U, V, t) \mapsto (-U, V, -t)$ and thus it admits a center on the local center manifold at the origin. $\square$

Proof of Theorem 6. We consider system (3) with the conditions of Theorem 6 that is we consider system

$$\begin{aligned}\dot{u} &= v,\\ \dot{v} &= -u + a_1u^2 + a_4v^2 + a_5vw + a_6w^2,\\ \dot{w} &= -w + c_1u^2 + c_2uv + c_3v^2.\end{aligned} \tag{24}$$

System (24) with the conditions of Theorem 6 (f.1) becomes

$$\begin{aligned}\dot{u} &= v,\\ \dot{v} &= -u + a_1u^2 + a_4v^2,\\ \dot{w} &= -w + c_1u^2 + c_2uv + c_3v^2.\end{aligned}$$

The center manifold is given by $w = \phi(u, v)$ for some analytic function ϕ. Restricting to this manifold we get that on the center manifold the dynamics is given by

$$\dot{u} = v, \qquad \dot{v} = -u + a_1u^2 + a_4v^2. \tag{25}$$

Note that this system is system (8) with $b = d = 0$, $a = a_1$, $c = a_4$, $A = 0$ and $B = -2a_4$. Hence, in view of Theorem 18 (ii) it has a center manifold at the origin.

System (24) with the conditions of Theorem 6 (f.2) becomes

$$\begin{aligned}\dot{u} &= v,\\ \dot{v} &= -u + a_1u^2 + a_4v^2 + a_6w^2\\ \dot{w} &= -w + c_1u^2 + 2c_1uv.\end{aligned} \tag{26}$$

Note that F given in (13) is an invariant algebraic surface for system (26) with cofactor $K = -1$. Since $F = 0$ is tangent to $w = 0$, it is a center manifold for system (26). Now we introduce the change of variables in (14). In these new variables system (26) becomes

$$\begin{aligned}\dot{U} &= V,\\ \dot{V} &= -U + a_1U^2 + a_4V^2 + a_6(W + c_1U^2)^2,\\ \dot{W} &= -W.\end{aligned}$$

The restriction on the center manifold $W = 0$ give rise to the system

$$\dot{U} = V, \qquad \dot{V} = -U + a_1 U^2 + a_4 V^2 + a_6 c_1^2 U^4.$$

This system is invariant by the symmetry $(U, V, t) \mapsto (U, -V, -t)$ and thus it has a center.

System (24) with the conditions of Theorem 6 (f.3) becomes

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= -u a_1 u^2 + a_4 v^2 + a_5 v w + a_6 w^2 \\ \dot{w} &= -w. \end{aligned} \tag{27}$$

It is clear that $w = 0$ is an invariant algebraic surface for system (27) with cofactor $K = -1$ and it is a center manifold for system (27). Restricting to the center manifold we get (25) and thus it has a center due to Theorem 18 (ii). ☐

Proof of Theorem 7. We consider system (3) with the conditions of Theorem 7 that is we consider system

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= -u + a_1 u^2 + a_3 u w + a_5 v w + a_6 w^2, \\ \dot{w} &= -w + c_1 u^2 + c_2 u v + c_3 v^2. \end{aligned} \tag{28}$$

Note that condition (g.3) is condition (f.1) of Theorem 6 thus we will not prove it. System (28) with the conditions of Theorem 7 (g.1) becomes

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= -u + a_1 u^2 + a_3 u w + a_6 w^2, \\ \dot{w} &= -w + c_1 u^2 + 2 c_1 u v. \end{aligned} \tag{29}$$

Note that F given in (13) is an invariant algebraic surface for system (29) with cofactor $K = -1$. Since $F = 0$ is tangent to $w = 0$, it is a center manifold for system (29). Now we introduce the change of variables in (14). In these new variables system (29) becomes

$$\begin{aligned} \dot{U} &= V, \\ \dot{V} &= -U + a_1 U^2 + a_3 U (W + c_1 U^2) + a_6 (W + c_1 U^2)^2, \\ \dot{W} &= -W. \end{aligned}$$

The restriction on the center manifold $W = 0$ give rise to the system

$$\dot{U} = V, \qquad \dot{V} = -U + a_1 U^2 + a_3 c_1 U^3 + a_6 c_1^2 U^4.$$

This system is invariant by the symmetry $(U, V, t) \mapsto (U, -V, -t)$ and thus it has a center.

System (28) with the conditions of Theorem 7 (g.2) becomes

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= -u + a_1 u^2 + a_3 u w + a_5 v w + a_6 w^2, \\ \dot{w} &= -w. \end{aligned} \tag{30}$$

It is clear that $w = 0$ is an invariant algebraic surface for system (30) with cofactor $K = -1$ and it is a center manifold for system (30). Restricting to the center manifold we get

$$\dot{u} = v, \qquad \dot{v} = -u + a_1 u^2,$$

which is system (25) with $a_4 = 0$. Hence, it has a center due to Theorem 18 (ii). □

Proof of Theorem 8. We consider system (3) with the conditions of Theorem 8 that is we consider system

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= -u + a_1 u^2 + a_3 uw + a_4 v^2 + a_6 w^2, \\ \dot{w} &= -w + c_1 u^2 + c_2 uv + c_3 v^2. \end{aligned} \tag{31}$$

Note that condition (h.1) is condition (f.1) of Theorem 6 and condition (h.2) is condition (a.1) of Theorem 1. Thus we will not prove them. System (31) with the conditions of Theorem 8 (h.3) becomes

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= -u + a_1 u^2 + a_3 uw + a_4 v^2 + a_6 w^2, \\ \dot{w} &= -w + c_1 u^2 + 2c_1 uv. \end{aligned} \tag{32}$$

Note that F given in (13) is an invariant algebraic surface for system (32) with cofactor $K = -1$. Since $F = 0$ is tangent to $w = 0$, it is a center manifold for system (32). Now we introduce the change of variables in (14). In these new variables system (32) becomes

$$\begin{aligned} \dot{U} &= V, \\ \dot{V} &= -U + a_1 U^2 + a_3 U(W + c_1 U^2) + a_4 V^2 + a_6 (W + c_1 U^2)^2, \\ \dot{W} &= -W. \end{aligned}$$

The restriction on the center manifold $W = 0$ give rise to the system

$$\dot{U} = V, \qquad \dot{V} = -U + a_1 U^2 + a_3 c_1 U^3 + a_4 V^2 + a_6 c_1^2 U^4.$$

This system is invariant by the symmetry $(U, V, t) \mapsto (U, -V, -t)$ and thus it has a center. □

Proof of Theorem 9. We consider system (3) with the conditions of Theorem 9 that is we consider system

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= -u + a_1 u^2 + a_3 uw + a_4 v^2 + a_5 vw, \\ \dot{w} &= -w + c_1 u^2 + c_2 uv + c_3 v^2. \end{aligned} \tag{33}$$

Note that condition (i.2) is condition (h.3) of Theorem 8 and condition (i.3) is condition (g.3) of Theorem 7. System (33) with the conditions of Theorem 8 (i.1) becomes

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= -u + a_1 u^2 + a_3 uw + a_4 v^2 + a_5 vw, \\ \dot{w} &= -w. \end{aligned} \tag{34}$$

It is clear that $w = 0$ is an invariant algebraic surface for system (34) with cofactor $K = -1$ and it is a center manifold for system (34). Restricting to the center manifold we get system (25) which has a center due to Theorem 18 (ii). □

Proof of Theorem 10. We consider system (3) with the conditions of Theorem 10 that is we consider system

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= -u + a_1 u^2 + a_2 uv + a_5 vw + a_6 w^2, \\ \dot{w} &= -w + c_1 u^2 + c_2 uv + c_3 v^2. \end{aligned} \tag{35}$$

Note that conditions (j.2) and (j.5) are conditions (b.1) and (b.5) of Theorem 2. Conditions (j.1) and (j.4) are conditions (f.3) and (f.1) of Theorem 6. Condition (j.6) is a particular case of condition (g.1) of Theorem 7. System (35) with the conditions of Theorem 10 (j.3) becomes

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= -u + a_1 u^2 + a_2 uv + a_5 vw, \\ \dot{w} &= -w + c_1 u^2 + 2c_1 uv \end{aligned} \tag{36}$$

with $2a_1a_2 + a_5c_2 = 0$. Note that F given in (13) is an invariant algebraic surface for system (36) with cofactor $K = -1$. Since $F = 0$ is tangent to $w = 0$, it is a center manifold for system (36). Now we introduce the change of variables in (14). In these new variables system (36) becomes

$$\begin{aligned} \dot{U} &= V, \\ \dot{V} &= -U + a_1 U^2 + a_2 UV + a_5 V(W + c_1 U^2), \\ \dot{W} &= -W. \end{aligned}$$

The restriction on the center manifold $W = 0$ give rise to the system

$$\dot{U} = V, \qquad \dot{V} = -U + a_1 U^2 + a_2 UV + a_5 c_1 V U^2, \tag{37}$$

with $2a_1a_2 + a_5c_2 = 0$. If $a_2 = 0$ we have $a_5c_2 = 0$. If $a_5 = 0$ then we obtain a particular case of case (j.4). If $c_2 = 0$ we are in the particular case (j.1). Hence we can assume $a_2 \neq 0$. In this case from the condition $2a_1a_2 + a_5c_2 = 0$ we isolate $a_1 = -a_5c_1/a_2$ recalling that $c_2 = 2c_1$. System (37) takes the form

$$\dot{U} = V, \qquad \dot{V} = -U - \frac{a_5 c_1}{a_2} U^2 + a_2 UV + a_5 c_1 V U^2. \tag{38}$$

System (38) has the inverse integrating factor $\mathcal{V} = 1 - a_2 V$. Hence, we have that $\mathcal{V}(0,0) \neq 0$ and consequently system (38) has a center in at the origin, see [7]. □

Proof of Theorem 11. We consider system (3) with the conditions of Theorem 11 that is we consider system

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= -u + a_1 u^2 + a_2 uv + a_4 v^2 + a_6 w^2, \\ \dot{w} &= -w + c_1 u^2 + c_2 uv + c_3 v^2. \end{aligned} \tag{39}$$

Note that conditions (k.1) and (k.2) are conditions (f.1) and (f.2) of Theorem 6. System (39) with the conditions of Theorem 11 (k.3) becomes

$$\begin{aligned}\dot{u} &= v,\\ \dot{v} &= -u + a_1u^2 + a_2uv - a_1v^2,\\ \dot{w} &= -w + c_1u^2 + c_2uv + c_3v^2.\end{aligned}$$

The center manifold is given by $w = \phi(u, v)$ for some analytic function ϕ. Restricting to this manifold we get that on the center manifold the dynamics is given by

$$\dot{u} = v, \qquad \dot{v} = -u + a_1u^2 + a_2uv - a_1v^2.$$

Note that this system is system (8) with $b = d = 0$, $a = -a_1$, $c = a_1$, $A = -a_2$ and $B = -2a_1$. Hence, in view of Theorem 18 (i) it has a center at the origin.

System (39) with the conditions of Theorem 11 (k.4) becomes

$$\begin{aligned}\dot{u} &= v,\\ \dot{v} &= -u + a_1u^2 + a_2uv - a_1v^2,\\ \dot{w} &= -w.\end{aligned} \tag{40}$$

It is clear that $w = 0$ is an invariant algebraic surface for system (40) with cofactor $K = -1$ and it is a center manifold for system (40). Restricting it to the center manifold we get

$$\dot{u} = v, \qquad \dot{v} = -u + a_1u^2 + a_2uv - a_1v^2, \tag{41}$$

which obviously has a center at the origin again due to Theorem 18 (i). This concludes the proof. □

Proof of Theorem 12. We consider system (3) with the conditions of Theorem 12 that is we consider system

$$\begin{aligned}\dot{u} &= v,\\ \dot{v} &= -u + a_1u^2 + a_2uv + a_4v^2 + a_5vw,\\ \dot{w} &= -w + c_1u^2 + c_2uv + c_3v^2.\end{aligned} \tag{42}$$

Note that conditions (l.1) and (l.3) are conditions (f.1) and (f.3) of Theorem 6, condition (l.2) is condition (k.3) of Theorem 11 and condition (l.5) is condition (j.3) of Theorem 10. System (42) with the conditions of Theorem 12 (l.4) becomes

$$\begin{aligned}\dot{u} &= v,\\ \dot{v} &= -u + a_1u^2 + a_2uv - a_1v^2 + a_5vw,\\ \dot{w} &= -w.\end{aligned} \tag{43}$$

It is clear that $w = 0$ is an invariant algebraic surface for system (43) with cofactor $K = -1$ and it is a center manifold for system (43). Restricting it to the center manifold we get (41) which has a center at the origin again due to Theorem 18 (i). □

Proof of Theorem 15. We consider system (3) with the conditions of Theorem 15 that is we consider system

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= -u + a_1u^2 + a_2uv + a_3uw + a_4v^2, \\ \dot{w} &= -w + c_1u^2 + c_2uv + c_3v^2. \end{aligned} \tag{44}$$

Note that conditions (o.1) and (o.4) are conditions (h.1) and (h.3) of Theorem 8, condition (o.2) is condition (k.3) of Theorem 11 and condition (o.5) is condition (c.2) of Theorem 3. System (44) with the conditions of Theorem 15 (o.3) becomes

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= -u + a_1u^2 + a_2uv + a_3uw - a_1v^2. \\ \dot{w} &= -w. \end{aligned} \tag{45}$$

It is clear that $w = 0$ is an invariant algebraic surface for system (45) with cofactor $K = -1$ and it is a center manifold for system (45). Restricting it to the center manifold we get (41) which has a center at the origin again due to Theorem 18 (i). □

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